Metric topology

Definition:

Let X be a non-empty set. Let d: X×X to R be a function. Suppose

1. $d(x,y) \geq 0$ for all x ,y €X
2. $d(x,y)=0$ x=y
3. $d(x,y)=d(y,x)$
4. $d(x,z) \leq d(x,y)+d(y,z)$ for all x ,y ,z €X

This d is called a metric or distance function. The set X is called a metric space. A metric space with a metric d is denoted by (X, d).

Definition:

Let (X ,d) be a metric space. Let $\varepsilon > 0$ be a positive real. Let x €X. Then we define $B_d(x,\varepsilon)=\{y/ d(x,y)<\varepsilon\}$. $B_d(x,\varepsilon)$ is called an £ ε- ball with centre x and radius ε.

Definition:

Let X be a non-empty set. Let d be a metric on X. Then the collection $b=\{B_d(x,\varepsilon)/x$ €X and $\varepsilon>0\}$ for a basis. The topology generated by b is called the metric topology induced by d.

Definition:

Let A be a subset of a metric space (X ,d). A is said to be bounded. If there exist a positive real M suchthat $d(x,y) \leq M$ for all x,y€Aor $d(a_1,a_2) \leq M$ for all a_1,a_2€A.

Boundedness of a set is not a topological property for it depends on the particular metric d that is used for X.

Definition:

Let (X, d) be a metric space. Let A contained X. Then the diameter of A is defined as diam A=sup{d(x,y)/x,y€A}.

Definition:

Let R denote the set of real number.

Consider $R^n= \{(x_1,x_2,\ldots,x_n)/x_i$€R for all i}

Let x€R^n.

$X=(x_1, x_2,\dots,x_n)$

Define $||x|| =(x_1^2+x_2^2+\dots+x_n^2)^{1/2}$

Let $x,y \in R^n$,we define $d(x,y)=||x-y||$

Then d is called n Euclidean metric of R^n.

$d(x,y)=||x-y||=(\text{summation } (x_i-y_i)^2)^{1/2}$.

The square matric f in R^n mis defined as $f(x,y)=\max\{|x_i-y_i|,i=1,2,\dots n\}$

Definition:

If X is a topological space X is called a metrizable if there exist a metric on the set X that induced to topology on X.

A metric space is a metrizable space X together with a specify metric d that gives the topology on X.

Theorem 20.1

Let X be a metric space with metric d. Define d :$X\times X$ to R by the equation $d(x,y)=\min\{d(x,y),1\}$.Then d is a metric that induces the same topology as d.

The metric d is called the standard bounded metric corresponding to d.

Proof:

Given (X, d) is a metric space.

Given $d(x, y)=\min\{d(x,y),1\}$

To prove d is metric

1. $d(x,y)=\min\{d(x,y),1\}\geq 0$
2. $d(x,y)=0,\min\{d(x,y),1\}=0$
   $d(x,y)=0$
   $x=y$
3. $d(x,y)=\min\{d(x,y),1\}$
   $=\min\{d(x, y), 1\}$
   $=d(y, x)$
4. To prove $d(x,z)\leq d(x,y)+d(y,z)$

Case 1:

Suppose $d(x, y)<1$ and $d(y,z)<1$

$d\ (x,y)=\min\{d(x,y),1\}=d(x,y)$

$d\ (y,z)=\min\{d(y,z),1\}=d(y,z)$

$d\ (x\ ,z)=\min\ \{d(x,z),1\}$

$d\ (x\ ,z)\leq d(x,y)+d(y,z)$

$\quad =d(x,\ y)+d(y,z)$

$d(x\ ,z)\leq d(x,y)+d(y,z)$

Case 2:

Suppose $d(x,\ y)<1,d(y,z)\geq 1$

$d(x,y)=\min\{d(x,y),1\}=d(x,y)$

$d(y,z)=\min\{d(x,z),1\}=1$

$d(x,y)+d(y,z)=d(x,y)+1$

$d(x,y)+d(y,z)\geq 1$

Now,$d(x,z)=\min\{d(x,z),1\}=1$

$d(x,z)\leq d(x,y)+d(y,z)$

Case 3:

Suppose $d(x,y)\geq 1$ and $d(y,z)\geq 1$

$d(x,y)=\min\{d(x,y),1\}=1$

$d(y,z)=\min\{d(y,z),1\}=1$

$d(x,y)+d(y,z)=2>1$

$d(x,y)+d(y,z)>1$

$d(x,z)=\min\ \{d(x,z),1\}$

$\quad =1$

$\quad <d(x,y)+d(y,z)$

$d(x,z)\leq d(x,y)+d(y,z)$

d is metric on X.

We note that in any metric space the collection of ε balls with $\varepsilon<1$ forms a basis for the metric topology for every basis element containing x contains such on £ balls centered at x.

It follows that d and d induced the same topology on X, because the collection of ε-balls with ε<1 under the two metrices are the same collection.

Lemma 20.2:

Let d and d' be two metrices on the set X.Let ỉ and ỉ' be the topologies they induce respectively .Then ỉ' is finer than ỉ iff for each x in X and each ε>0 there exists a δ>0 suchthat $B_d(x,\delta)$ contained in $b_d(x,\varepsilon)$.

Let x€X.

Let ε>0 be given.

Consider $B_d(x,\varepsilon)$

Clearly x€$B_d(x,\varepsilon)$ there exist B'€b ,B' is a basis element ,B' is open then there exist δ>0 suchthat

x €$B_d(x ,\delta)$ contained in B' C $B_d(x,\varepsilon)$

$B_d(x, \delta)$ C $B_d(x, \varepsilon)$.

Conversely,

Given $B_d(x, \delta)$ C$B_d(x,\varepsilon)$

To prove ỉ' is finer than ỉ.

Let x€X.

Let B be the basis element containing X.

x€B.

Then there exist ε>0 suchthat x€$B_d(x,\varepsilon)$CB from the given there exist δ>0 suchthat x€$B_{d'}(x,\delta)$ C$B_d(x,\varepsilon)$CB.

Put B'=$B_d(x,\delta)$.

x€B'CB

There exist B'€b' suchthat x€B'CB.

Therefore ỉ' is finer than ỉ.

METRIC TOPOLOGY (CONTINUED)

Theorem:21.1

Let $f:X \rightarrow Y$ let X and Y be metrizable with metrices d_x and d_y respectively then continunity of f is equivalent to the requirements that given $x \in X$ and given $\varepsilon>0$ there exist $\delta>0$ such that $d_x(x,y)<\delta \Rightarrow d_y(f(x),f(y))< \varepsilon$

Proof:

Given that (x,d_x) and (Y,d_y) are two metric space

To prove :given $\varepsilon >0$ there exist $\delta>0$ such that $d_x(x,y)<\delta \Rightarrow d_y(f(x),f(y))< \varepsilon$

Let $x \in X$

Let $\varepsilon >0$ be given

Consider $B_{dy} (f(x), \varepsilon)$ is open in y

Since f is continuous, $f^{-1}(B_{dy} (f(x), \varepsilon))$ is open in x

Clearly $x \in f^{-1}(B_{dy} (f(x), \varepsilon))$

Therefore , there exist $\delta>0$ such that $x \in B_{dx}(x, \delta) \subset f^{-1}(B_{dy} (f(x), \varepsilon))$

Now, $d_x(x,y)<\delta \Rightarrow y \in B_d(x, \delta)$

$\Rightarrow y \in f^{-1}(B_{dy} (f(x), \varepsilon))$

$\Rightarrow f(y) \in B_d(f(x), \varepsilon)$

$\Rightarrow d_y(f(x),f(y))< \varepsilon$

Conversely, assume that given $x \in X$ and given $\varepsilon >0$ there exist $\delta>0$ such that $d_x(x,y)<\delta \Rightarrow d_y(f(x),f(y))< \varepsilon$

To prove : f is continuous

Let v be a open in y containing a point of f(x)

ie,$f(x) \in V$,there exist $\varepsilon >0$ such that $f(x) \in (B_{dy} (f(x), \varepsilon)) \subset V$

$\Rightarrow x \in f^{-1}(B_{dy} (f(x), \varepsilon)) \subset f^{-1} (v)$- - - - (1)

We have $d_x(x,y)<\delta \Rightarrow d_y(f(x),f(y))< \varepsilon$

Let $y \in B_{dx}(x, \delta)$

$\Rightarrow d_x(x,y)<\delta$

$\Rightarrow d_y(f(x),f(y))< \varepsilon$

$\Rightarrow f(y) \in B_{dy} (f(x), \varepsilon)$

$Y \in f^{-1}(B_{dy} (f(x), \varepsilon)$

$B_{dx}(x, \delta) \subset f^{-1}(B_{dy} (f(x), \varepsilon)$- - - - - - - (2)

From (1) and (2)

$x \in B_{dx}(x, \delta) \subset f^{-1}(B_{dy} (f(x), \varepsilon) \subset f^{-1}(v)$

Therefore $x \in B_{dx}(x, \delta) \subset f^{-1}(v)$

Therefore $f^{-1}(v)$ is open in X

Therefore f is continuous

Definition:

let $x_1,x_2,$..........be the sequences of points in a topological space X. It is said to be converge to a point $x \in X$ iff for all neighbourhood U of X there is a positive integer N such that $x_n \in U$ for all $n \geq N$.This is denoted by $(x_n)\to x$

Lemma :21.2 (The sequence lemma)

Let X be a topological space .let $A \subset X$.If there is a sequences of points of A converge to x then $X \in \bar{A}$.The converse holds if X is a metrizable

Proof:

Let X be a topological space and $A \subset X$

Suppose there is a sequence of points of A converging to x

To prove: $X \in \bar{A}$

Since $(x_n)\to x$, for all neighbourhood U of x there exist N such that $x_n \in U$ for all $n \geq N$

Therefore , $x_n \in A$

Therefore ,U intersect A , for all neighbourhood U containing x intersects A

Therefore , $X \in \bar{A}$

Conversely, given that X is metrizable and $X \in \bar{A}$

Let d be a metric for the topology of X

To prove: there exist (X_n) ЄA such that $(X_n)\rightarrow X$

Consider B(x,1)

B(x,1) is an open set containing x

Since XЄĀ, $B_d(x,1)\cap A\neq\Phi$

Let x_1 Є $B_d(x,1)\cap A$

Consider B(x,1/2)

This is an open set containing x

$B_d(x,1/2)\cap A\neq\Phi$

Let x_2Є $B_d(x,1/2)\cap A$

………..……….

x_n Є $B_d(x,1/2)\cap A$

…………………..

Clearly (X_n) ЄA--------------(1)

To prove : $(X_n)\rightarrow x$

Let U be a neighbourhood of x ,x ЄU there exist ε>0 such that xЄB(x,ε)CU

Choose N such that 1/N<ε

w.k.t, x_N Є $B_d(x,1/N)$

$\Rightarrow d(x,x_N)<\frac{1}{N}<\varepsilon$

$\Rightarrow x_{N+1}$ Є$B_d(x,\frac{1}{N+1})$

$\Rightarrow d(x,x_{N+1})<\frac{1}{N+1} < \frac{1}{N} < \varepsilon$

$d(x,x_{N+1})< \varepsilon$

………

x_nЄ$B_d(x,1/n)$

$\Rightarrow d(x,x_n)<1/n<1/N<\varepsilon$ for all $n\geq N$

$\Rightarrow d(x,x_n)<\varepsilon$ for all $n\geq N$

Therefore $x_n \in B_d(x,\varepsilon) \subset U$ for all $n\geq N$

$x_n \in U$ for all $n\geq N$

for all neighbourhood U of x there exist U of X there exist N such that $x_n \in U$ for all $n\geq N$

Therefore,$(x_n)\rightarrow x$---------(2)

From(1) and (2),

There exist $(x_n)\in A$ such that $(x_n)\rightarrow x$

Theorem :21.3

Let $f:X\rightarrow Y$.If the function f is continuous then for every convergences sequences $(x_n)\rightarrow x$ in X,the sequences $(f(x_n))\rightarrow f(x)$.The converse holds if X is metrizable

(or)

Let $f:X\rightarrow Y$.Let X be metrizable .The function f is continuous iff for all $(x_n)\rightarrow x$ in X

$\Rightarrow (f(x_n))\rightarrow f(x)$ in Y

Proof:

Let $f:X\rightarrow Y$.Let X be metrizable

Suppose $f:X\rightarrow Y$is continuous

To prove $(x_n)\rightarrow x$ in X

$\Rightarrow (f(x_n))\rightarrow f(x)$ in Y

Let v be a neighbourhood of f(x) in Y,$f(x)\in V\Rightarrow x\in f^{-1}(v)$

Since v is open in Y and $f:X\rightarrow Y$is continuous, $f^{-1}(v)$ is open in x

Now, since $(x_n)\rightarrow x$,there exist N such that $x_n\in f^{-1}(v)$ for all $n\geq N$

Therefore $f(x_n)\in V$,for all $n\geq N$

Therefore , $(f(x_n))\rightarrow f(x)$ in Y

Conversely,suppose $(x_n)\rightarrow x$ in $X\Rightarrow (f(x_n))\rightarrow f(x)$ in Y

To prove : $f:X\to Y$is continuous

Let A be a subset of X

To prove: $f(\bar{A}) C f(A)$

Let $f(x) \in f(\bar{A})$

$=> x \in \bar{A}$

By sequence lemma ,there exist $x_n \in A$ such that $(x_n)\to x$

Therefore $f(x) \in f(\bar{A})$

Therefore $f(\bar{A}) C f(A)$

Therefore f is continuous .

Definition :

A space X is said to have a countable basis at the point x if there is s countable collection $\{U_n\}_{n\in z^+}$ of neighbourhood of x such that any neighbourhood U of x contains atleast one of the sets U_n.

The space X has a countable basis at each of its point is said to satisfy the first countability axiom .

Theorem : 21.5

If X is a topological space and if $f,g:X\to R$ are continuous function then f+g,f-g,f.g are continuous .If $g(x)\neq o$ for all ,then f/g is continuous.

Proof:

The map $h:X\to R\times R$ defined,

$h(x)=f(x)\times g(x)$ is continuous .

The function f+g equals the composite of h and the addition operation $,+:R\times R\to R$

therefore , f+g is continuous.

Similarly, f-g equals the composite of h and the subtraction operation $,-:R\times R\to R$

therefore , f-g is continuous.

And , f.g equals the composite of h and the multiplication operation, $.:R\times R\to R$

therefore , f.g is continuous.

And, f/g equals the composite of h and the division operation , $/:R\times R\rightarrow R$

therefore , f/g is continuous.

Definition :

Let $f_n:X\rightarrow Y$ be a sequences of function from the set X to the metric space Y . Let d be the metric for Y.We say that the sequence(f_n) converges uniformly to the function f: $X\rightarrow Y$,if given $\varepsilon>0$ there exist an integer N such that $d(f_n(x),f(x))<\varepsilon$ for all $n\geq N$ and all x in X .Uniformly of converges depends not only on the topology of Y but also on its metric.